The Alexander Polynomial

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Algebraic Topology, Fall 2022

September 30, 2023

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1 Introduction and motivation

Fundamental theorems and constructions in Algebraic Topology often arise by examining mappings of n-spheres into the spaces we want to study. By adopting a group theoretic lens, we are able to define space-distinguishing invariants, including homotopy and (co)homology. Simultaneously, we often define the basic theories of (co)homology by decomposing spaces into easily understood pieces, and gluing them back up in some prescribed way. Despite choices being made, many of these (co)homology theories are independent of the decomposition, and so do indeed define isomorphic invariants. The study of knots draws inspiration from both of these perspectives, and asks the question

"Given two embeddings of the circle into some 3-dimensional space, to what extent are the embedded loops the same?"

The answer to this question has deep implications for 3-manifold theory. Any (suitably nice) compact, oriented 3-manifold can be described as the glued union of two handlebodies, where the gluing is specified uniquely by identifying knots on their surfaces. So there is certainly value in establishing *what* it means for two knots to be the same, and *how* we can tell them apart given this definition.

The classification of knots has been an active area of research for the last century and a half. In 1926, one of the first instances of substantive progress was made by James Waddell Alexander, who introduced an integral Laurent polynomial arising from a certain fundamental group associated to a knot [Ale28]. This polynomial proved to be invariant under isotopy, thus defining a knot invariant. The Alexander polynomial remained the only known knot polynomial until the discovery of the Jones polynomial [Jon85]. Since then, several other knot polynomials and invariants have emerged, each with its own merit against the others. Though antiquated, the Alexander polynomial remains an important knot invariant, which can be easily understood using only basic algebraic topology constructions.

We offer a comprehensive guide to understanding the Alexander polynomial, first investigating the geometric origins of the invariant via a cellular structure of the knot complement, and then enhancing the method of calculation by some algebraic observations. We conclude the exposition by highlighting some key properties of the Alexander polynomial.

1.1 A note on sources

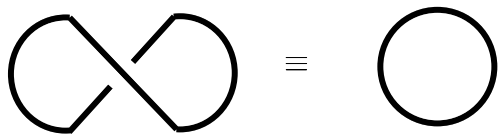
There are plenty of wonderful resources for studying knots. The canonical guide is Rolfsen's *Knots and Links* [Rol90], in particular chapters 3 and 7. The referenced work is a revised version of the original which came out in the 70s. I find the writing a bit outdated, and much prefer Lickorish's *An Introduction to Knot Theory*. The key concepts can be found in chapters 1, 5, 6 and 7, though the rest of the textbook works to supplement the material in these sections and so is worth a read. Besides these two textbooks, there are plenty of sets of notes online written by lecturers and students alike that are easily accessible and relatively accurate. Two reliable sets are Kauffman's essay *Knot Theory* [Kau16] and Rasmussen's lecture notes *Knots, polynomials, and categorification* [Ras21]. For the sake of avoiding repetition, I will preface this essay by saying that its contents implicitly cites all of the above as a source. The basic definitions, theorems and proofs all can be found within these sources, and so I will not make further mention of their origins in the essay. I will however explicitly cite sources whose perspectives I adopt for specific sections.

2 Knots, links, and complements

2.1 Basic definitions

Definition 2.1. A *knot* in $\mathbb{R}^3$ is an embedding of a loop $K : S^1 \hookrightarrow \mathbb{R}^3$. By embedding, we mean either continuous, piecewise-linear or smooth, the three being interchangeable in the context of 3-dimensional topology. A *link* in $\mathbb{R}^3$ is an embedding of finitely many disjoint loops $L : \sqcup_n S^1 \hookrightarrow \mathbb{R}^3$.

In some respect I've already lied to you - by an embedding, I should really say a class of embeddings that are *isotopic* to each other. This helps make sure we remember that things like the "infinity loop" and the *unknot*, are really just the same, and that we can get from one to the other by a smooth family of diffeomorphisms.



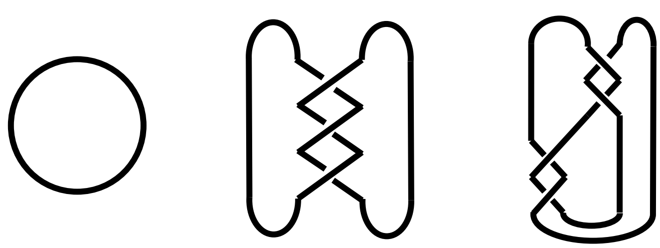
Also, its not strictly true that we *have* to map into $\mathbb{R}^3$. In fact, thinking about embeddings into other 3-dimensional spaces has lead to incredibly fruitful research and the invention of entirely new branches of topology. Moreover, it is often more useful to think about knots embedded in S^3 and not $\mathbb{R}^3$, as will be seen. We can think of these two codomains interchangeably though for dimension reasons. While we're at it, it's best to decide how we'd like to represent a knot in two dimensions, so that it's easy to tell when they are the same. The simplest way to do this is to imagine a knot hovering in three dimensions, and then squash it straight down onto the floor. Looking directly down at the knot gives us a 2-dimensional picture, while still giving us enough information about how a point would move along the loop at each "crossing".

Definition 2.2. Formally, we represent a knot in $\mathbb{R}^2$ by a *knot diagram*, which comprises

1. a smooth map $\varphi : S^1 \hookrightarrow \mathbb{R}^2$ such that $\varphi'(\theta) \neq 0$ for all $\theta \in S^1$, $\varphi(S^1)$ has no transverse intersections, and $\varphi(S^1)$ has no triple points, along with

2. an ordering of each pair of intersection points.

For any knot K in $\mathbb{R}^3$ or S^3, there is an open, dense subset of 2-planes such that projecting onto one of them gives a knot diagram. So throwing a knot against any surface is likely to give a valid knot diagram. Yay!



(a) (b) (c)

Figure 1: Examples of knot diagrams of (a) the unknot, (b) the (positive) trefoil, and (c) the figure-8 knot.

2.2 Link complements

Knots and links in S^3 really only take on any meaning when in the context of their ambient spaces, so it is reasonable to try to characterise a knot K by its corresponding *knot complement* $M_K := S^3 \setminus K$.

Warning!. Remember when I lied to you? A knot is not just a single embedding, rather an isotopy class of embeddings. We need to make sure that our definition of the knot complement is compatible with this! The next lemma asserts exactly that this is the case.

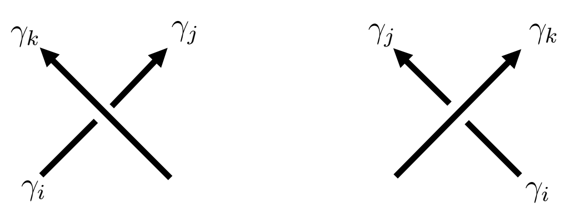
Lemma 2.3. Knots K_1 and K_2 in S^3 are isotopic if and only if their knot complements are orientation-preserving diffeomorphic.

Proof. That orientation-preserving diffeomorphic knot complements give isotopic knots is a difficult theorem due to Gordon and Leucke, see [GL89]. The converse is not as hard to prove. Let $j_1, j_2 : S^1 \hookrightarrow S^3$ be two isotopic knots embedded in S^3. Then by definition of isotopy, there exists a family of diffeomorphisms $F : S^1 \times [0,1] \to S^3$ such that $F_0 = j_1$ and $F_1 = j_2$. But this isotopy is enough to tell us exactly how to mould M_{K_1} into M_{K_2}. Define a time-dependent vector field on $F(S^1 \times [0,1])$ by pushing forward $\frac{\partial}{\partial t}$, $v_t := df\left(\frac{\partial}{\partial t}\right)$. We can extend v_t to a vector field on S^3 smoothly, which has an associated global flow $f_t : S^3 \to S^3$. But this map gives us exactly what we need: the map $f_1 : M_{K_0} \mapsto M_{K_1}$ is an orientation-preserving diffeomorphism between M_{K_0} and M_{K_1}. $\square$

3 The fundamental group of a knot complement

With the knot complement certifiably well-defined, let's crack into possible invariants that arise by examining it. The most basic invariant is its fundamental group, which we often refer to as *the knot group*. It turns out that we can calculate it explicitly using what's known as the *Dehn presentation* or the *Wirtinger presentation*. Of course the methods give isomorphic groups, but the Wirtinger presentation is usually cleaner since generally it will always spit out one more relation than generator. The method is outlined below, and we offer a sketch of the proof that it is isomorphic to the fundamental group of M_K.

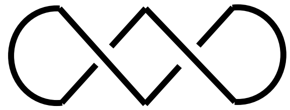
Definition 3.1. Let K be a knot in S^3, and let D be a knot diagram for K. Equip D with an (arbitrary) orientation. Then the *Wirtinger presentation* of M_K is a presentation of the fundamental group $\pi_1(M_K)$ with generators specified by the arcs of D, and relations given by conjugation according crossings. Specifically, a relation in $\pi_1(M_K)$ corresponds to a crossing by the following rule:



$\gamma_j = \gamma_k \gamma_i \gamma_k^{-1}$ $\qquad$ $\gamma_j = \gamma_k^{-1} \gamma_i \gamma_k$

Before proving why this is in fact a presentation of $\pi_1(M_K)$, let's look at some examples first.

Example 3.2 (Wirtinger Presentation of a nontrivial presentation of the unknot)**.** Consider the following knot diagram of the unknot:

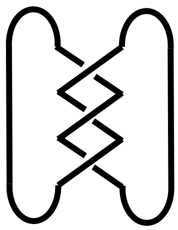


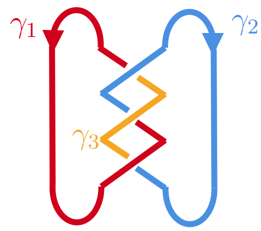
In the above presentation, there are two generators γ_1 and γ_2 corresponding to the left and right arcs respectively, and the relations following the above rule are $\gamma_2 = \gamma_1\gamma_1\gamma_1^{-1}$ and $\gamma_1 = \gamma_2\gamma_2\gamma_2^{-1}$. These relations both say that $\gamma_1 = \gamma_2$, so that

$$\pi_1(M_K) = \langle \gamma_1, \gamma_2 \mid \gamma_1 = \gamma_2 \rangle = \langle \gamma_1 \rangle,$$

which is unsurprising since the unknot complement is diffeomorphic to a solid 2-torus.

Example 3.3 (Wirtinger Presentation of trefoil)**.** Consider the following diagram of the (positive) trefoil: There are three generators, γ_1, γ_2 and γ_3, and three relations:



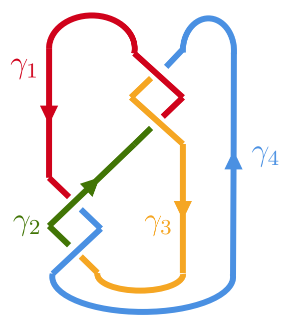


$$\gamma_1 = \gamma_2^{-1}\gamma_3\gamma_2, \quad \gamma_2 = \gamma_3^{-1}\gamma_1\gamma_3, \quad \text{and} \quad \gamma_3 = \gamma_1^{-1}\gamma_2\gamma_1.$$

Substituting the third relation into the first and second give the same relation, so that the knot group is

$$\pi_1(M_K) = \langle \gamma_1, \gamma_2 \mid \gamma_2 - 1\gamma_1^{-1}\gamma_2^{-1}\gamma_1\gamma_2\gamma_1 \rangle.$$

Example 3.4 (Wirtinger Presentation of the figure-8 knot)**.** We'll make use of our previous presentation of the figure-8 knot from before:



We have four arcs, $\gamma_1, \gamma_2, \gamma_3$ and γ_4 which generate $\pi_1(M_K)$. There are also four crossings, which give rise to the following relations:

$$\gamma_4 = \gamma_2\gamma_1\gamma_2^{-1}, \quad \gamma_2 = \gamma_4\gamma_3\gamma_4^{-1}, \quad \gamma_3 = \gamma_1^{-1}\gamma_4\gamma_1, \quad \text{and} \quad \gamma_1 = \gamma_3^{-1}\gamma_2\gamma_3.$$

Since we have four generators and four relations, we can throw the last of them away and combine the rest to give

$$\pi_1(M_K) = \langle \gamma_1, \gamma_2 \mid \gamma_2\gamma_1\gamma_2^{-1}\gamma_1^{-1}\gamma_2\gamma_1\gamma_2^{-1}\gamma_1\gamma_2^{-1}\gamma_1^{-1} \rangle.$$

Proposition 3.5. The group defined in definition 3.1 is isomorphic to $\pi_1(M_K)$.

Sketch of proof. The following sketch of a proof can be formalised via Van Kampen's theorem, constructing a 2-dimensional cell complex that is a deformation retract of the knot complement, which has the Wirtinger presentation as its fundamental group. Geometrically though, it is intuitively clear where the generators and relations come from. The generators correspond to loops based at infinity, each passing under an arc of the disconnected knot diagram. The relations come from how we can homotope these loops across crossings. These are both represented in fig. 2.

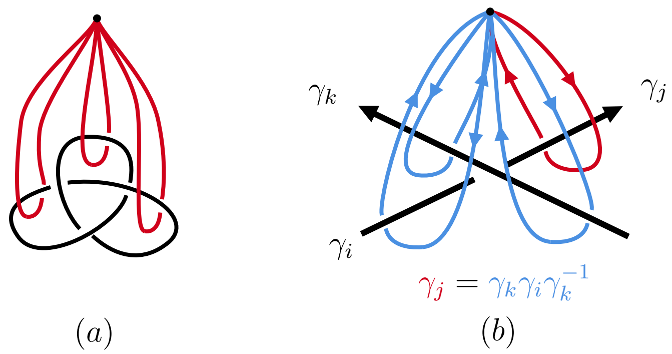


Figure 2: The origin of (a) generators and (b) relations in the Wirtinger presentation

□

Now that we know how to calculate the knot group exactly, we may wish to explore other potential avenues for knot invariants. The next logical step would be to see what we can discern from the homology groups of the knot complement. Unfortunately they turn out to be much less informative than homotopy groups. In fact, they are so less informative that they says absolutely nothing about the knot in question.

Proposition 3.6. The first homology group of the knot complement is a terrible invariant.

Proof. Consider a closed tubular neighbourhood $N(K)$ of $K \subset S^3$, then $S^3 \setminus K$ is homotopic to $S^3 \setminus N(K)$. In particular, $H_1(S^3 \setminus K) \simeq H_1(S^3 \setminus N(K))$, but $S^3 \setminus N(K)$ is homotopic to the solid 2-torus T^2. Thus $H_1(S^3 \setminus K) \simeq \mathbb{Z}$, so first homology captures no information about K. □

From a group-theoretic standpoint, it is incredibly difficult to tell when two knots are isomorphic by just observing their knot groups. Abelian groups are often easier to deal with, but our above calculation suggests that we've hit a wall with homology. Or have we?

4 The Alexander polynomial

It turns out that digging a little deeper into *how* the homology groups and the knot group are related gives rise to a new invariant, the *Alexander polynomial*. There are a variety of ways of thinking about the Alexander polynomial, and the sources mentioned at the very beginning of this essay exhibit a good chunk of them. The most natural way to define the polynomial is by adopting a geometric perspective while exploiting some basic algebraic theory of covering spaces. A good resource for this is chapter 2 of Rasmussen's notes *Knots, polynomials, and categorification* [Ras21]. The arguments of this section will follow this source closely, but the description of the *Fox calculus* is more reminiscent of that that appears in chapter 11 of Lickorish's *An Introduction to Knot Theory* [Lic97]. Thus, we'll begin by describing the Alexander polynomial via a geometric perspective. We'll then develop an algebraic interpretation of the Alexander polynomial, which makes its calculation much more efficient.

4.1 A geometric perspective

Denote the abelianisation map by $|\cdot| : \pi_1(M_K) \to H_1(M_K) \simeq \mathbb{Z}\langle t \rangle$. Since $H_1(M_K)$ does not tell us much about K, its reasonable to lift M_K to a covering space $\overline{M_K}$, and examine its first homology. The idea is that perhaps we can reach a middle ground somewhere between M_K and its universal covering space, where both π_1 and H_1 are nontrivial. By the Galois theory of covering spaces, any subgroup of $\pi_1(M_K)$ corresponds to a covering space of M_K. The most natural subgroup to consider is $\ker|\cdot|$, so lets do that.

Definition 4.1 (Infinite cyclic cover)**.** The covering space $p : \overline{M_K} \to M_K$ of M_K corresponding to the subgroup $\ker|\cdot| \leq \pi_1(M_K)$ is called the *infinite cyclic cover of* M_K, and has deck group $\pi_1(M_K)/\ker|\cdot| \simeq \mathbb{Z}\langle t \rangle$.

We'd like to examine the action of a deck transformation acting on $\overline{M_K}$, but in order to do so we need to understand the geometry of M_K and thus of $\overline{M_K}$.

Proposition 4.2. The knot complement M_K has the structure of a 3-dimensional handlebody. The handle decomposition is described by a presentation of its fundamental group.

Note of proof. Suppose M_K has fundamental group presentation

$$\pi_1(M_K) = \langle \gamma_1, ... \gamma_n \mid w_1, ..., w_m . \rangle.$$

Then M_K is homotopic to a cell complex with one 0-cell, n 1-cells corresponding to the generators, and m 2-cells attached to the 1-skeleton via boundary maps specified by the relations. A good visual of this is offered in [Car12], see Figures 19 and 20. □

Example 4.3 (Infinite cyclic cover of trefoil knot)**.** It is best to understand the cellular decomposition of the knot complement and consequently the infinite cyclic cover via an example. It is a quick check to show that the fundamental group of the trefoil is given by

$$\pi_1(M_K) = \langle \alpha, \beta \mid \alpha\beta\alpha\beta^{-1}\alpha^{-1}\beta^{-1} \rangle,$$

and so M_K is homotopic to a 2-dimensional cell complex with one 0-cell p, two 1-cells α and β, and one 2-cell w with attaching circle given by $\alpha\beta\alpha\beta^{-1}\alpha^{-1}\beta^{-1}$.

With a cell-complex structure of (a deformation retract of) the knot complement, it is easy find a cell-complex structure of $\overline{M_K}$. Choose a lift of p into $\overline{M_K}$, and denote this 0-cell by $\overline{p}$. Then since the action of the deck group (generated by t) is transitive, $\overline{M_K}$ has 0-cells $t^n \cdot \overline{p}$, $n \in \mathbb{Z}$. Similarly, we can lift α and β, and arrive at a 1-skeleton that for our example looks like

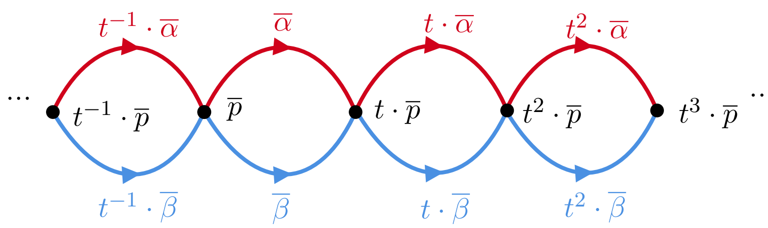


Figure 3: 1-cell structure of $\overline{M_K}$ for K the trefoil.

Note that the boundary map of the 2-cell w lifts to a closed curve in $\overline{M_K}$, describing the attaching map of one of the 2-cells in $\overline{M_K}$. The rest of the 2-cells are obtained via deck transformations.

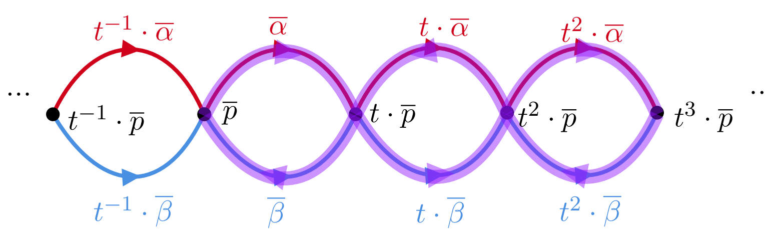


Figure 4: Attaching circle of $\overline{w}$ in $\overline{M_K}$.

The goal is to use this cell complex structure to analyse $H_1(\overline{M_K})$, in the hopes that it is nontrivial and thus harbours some information about K. We'll see that in fact $H_1(\overline{M_K})$ has a nice algebraic description as a module over the integral ring of Laurent polynomials, arising from the action of the deck group. This is clear in our example of the trefoil, where the cellular chain complex of $\overline{M_K}$ is a module over $R := \mathbb{Z}[\langle t \rangle] = \mathbb{Z}[t^{\pm 1}]$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \longrightarrow & R \oplus R & \longrightarrow & R & \longrightarrow & 0 \\ & & \langle \overline{w} \rangle & & \langle \overline{\alpha}, \overline{\beta} \rangle & & \langle \overline{p} \rangle & & \end{array}$$

Definition 4.4 (Alexander module)**.** Let K be a knot, M_K be its knot complement, and $\overline{M_K}$ be the infinite cyclic cover. Then we define the *Alexander module* of K to be the group $H_1(\overline{M_K})$, seen as a module over $\mathbb{Z}[t^{\pm 1}]$, where t represents a generator of the deck group.

Switching coefficients from $\mathbb{Z}$ to $\mathbb{Q}$ allows us to do further algebraic manoeuvres, since our module becomes one over a principal ideal domain. This allow us to represent the Alexander module as a polynomial via the Structure theorem for finitely generated modules over a PID. From now on, we unambiguously refer to the ring of Laurent polynomials over $\mathbb{Q}$ as R. We can decompose

$$H_1(\overline{M_K}, \mathbb{Q}) = R^k \oplus {R}/{p_1} \oplus \cdots \oplus {R}/{p_l},$$

where $p_1, ..p_l$ are some Laurent polynomials $\in R$. Picking apart the relationship between M_K and $\overline{M_K}$'s chain complexes tells us that actually $k = 0$, so that $H_1(M_K, \mathbb{Q})$ is a torsion module. We can thus define an invariant of K to be the product of all the polynomials $p_1 ... p_l$:

Definition 4.5 (Alexander polynomial)**.** Given a decomposition

$$H_1(\overline{M_K}, \mathbb{Q}) = {}^{R}\!/_{p_1} \oplus \cdots \oplus {}^{R}\!/_{p_l},$$

we define the *Alexander polynomial* of K to be the Laurent polynomial

$$\Delta_K(t) := \prod_{i=1}^{l} p_i.$$

Before we wrap up this section by finishing our example of the trefoil knot, let's take a minute to make a couple of important remarks. Immediately obvious is the fact that the Alexander polynomial is **not** unique, insofar as it is only well-defined up to multiplication by a unit in R. So we only speak of a *representative* of the Alexander polynomial, and we denote equivalence by $\sim$. It is also worth pointing out that interpreting the Alexander polynomial geometrically can be rather laborious, as you really need to understand the underlying structure of the knot complement. In the case of the trefoil or other basic knots, it is easy to understand the action of the deck group on lifts of cells, but for more complicated ones (or even links!) it becomes way trickier. We will see soon that taking a more algebraic perspective allows us to mitigate this difficulty.
Finally, let us reflect on why exactly the Alexander polynomial is of use to us in comparison to the fundamental group. We remarked in the previous section that, though the fundamental group is a strong invariant, it is very difficult to use when needing to tell whether two knots are the same or different. A group has an infinite number of presentations, and there is no easy way to go from one to another. On the other hand, the Alexander polynomial is incredibly easy to identify, since all we need to do to determine whether two knots are different is check that they do not differ by some multiplicative unit in R.

Example 4.6 (Alexander polynomial of trefoil)**.** Let's finally calculate the Alexander polynomial of the trefoil. We know that the cell chain complex of $\overline{M_K}$ looks like

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \longrightarrow & R \oplus R & \longrightarrow & R & \longrightarrow & 0 \\ & & \langle \overline{w} \rangle & & \langle \overline{\alpha}, \overline{\beta} \rangle & & \langle \overline{p} \rangle & & \end{array}$$

The differentials are also not difficult to deduce. From fig. 3 and fig. 4, it is clear that $d_1(\overline{\alpha}) = d_1(\overline{\beta}) = t \cdot \overline{p} - \overline{p} = (t-1)\overline{p}$, so that d_1 is represented by the matrix $d_1 = [t-1 \; t-1]$. The map d_2 is also fairly straightforward to calculate. Following the boundary of fig. 4, we have that

$$\begin{aligned} d_2(\overline{w}) &= \overline{\alpha} + t \cdot \overline{\beta} + t^2 \cdot \overline{\alpha} - t^2 \cdot \overline{\beta} - t \cdot \alpha - \overline{\beta}, \\ &= (\overline{\alpha} - \overline{\beta})(t^2 - t + 1). \end{aligned}$$

Hence $\ker(d_1) = \langle (-1, 1) \rangle$, and $\operatorname{Im}(d_2) = \langle (t^2 - t + 1, -t^2 + t - 1) \rangle$, so that

$$H_1(\overline{M_K}) = {}^{\langle (-1,1) \rangle}\!/_{\langle (t^2 - t + 1, -t^2 + t - 1) \rangle} \simeq {}^{R}\!/_{(t^2 - t + 1)}.$$

Thus the Alexander polynomial of the trefoil is

$$\Delta_K(t) \sim t^2 - t + 1.$$

4.2 An algebraic observation

The Alexander module is not just a construction that exists for knot groups, but for any group with a specified presentation and satisfying the condition that its abelianisation is $\mathbb{Z}$.
Suppose we have some commutative unital ring R, and a module M over R.

Definition 4.7. We say that M is finitely presented if we have an exact sequence

$$0 \to F \xrightarrow{p} E \xrightarrow{\pi} M \to 0.$$

The exact sequence is known as a presentation of M.

This looks awfully familiar to the chain complex we described for the complement of the trefoil! In fact, any chain complex of a knot complement has this form, and so we can translate our geometric picture into an algebraic sequence. We'll show how the presentation matrix of the complex contains all the information we need to determine the Alexander polynomial.

Proposition 4.8. Given a knot K and a presentation of its knot group

$$\pi_1(M_K) = \langle \gamma_1, ..., \gamma_n \mid w_1, ..., w_m \rangle,$$

the differential maps in the chain complex

$$\begin{array}{ccccccc} 0 \longrightarrow & R^m & \xrightarrow{d_2} & R^n & \xrightarrow{d_1} & R & \longrightarrow 0 \\ & \langle \overline{w_1}, ..., \overline{w_m} \rangle & & \langle \overline{\gamma_1}, ..., \overline{\gamma_n} \rangle & & \langle \overline{p} \rangle & \end{array}$$

can be calculated in a systematic way via *Fox Calculus*. Explicitly, if $|\cdot| : \pi_1(M_K) \to H_1(M_K)$ denotes the abelianisation map, then

1. the map d_1 is given by $d_1(\gamma_i) = (|\gamma_i| - 1)\overline{p}$, and
2. the map d_2 is given by the following free differential: for w_i a relation,

$$d_2(w_i) = \sum_j |\phi(d_{\gamma_j} w_i)| \cdot \overline{\gamma_j},$$

where $\phi : \mathbb{Z}[F] \to \mathbb{Z}[\pi_1(M_k)]$ is the canonical map of rings where F is the free group on m generators, $F = \langle w_1, ..., w_m \rangle$, and d_{γ_j} acting on $\mathbb{Z}[F]$ is the linear extension of the map on F defined by

$$d_{\gamma_j}\gamma_i = \delta^i_j, \quad d_{\gamma_j}\gamma_i^{-1} = -\delta^i_j\gamma_i^{-1}, \quad \text{and} \quad d_{\gamma_j}(ww') = d_{\gamma_j}w + w d_{\gamma_j}w'.$$

Proof. The map d_1 takes any generator γ_i and maps it to its signed boundary, which is $|\gamma_i|\overline{p} - \overline{p}$. This is clear from the geometric pictures we had before. The map d_2 is trickier to understand, since we're now dealing with a map from $R^m \to R^n$ and so it has more moving parts. Geometrically, for each γ_j the jith entry of the matrix representing this map essentially gives a signed count of the segments of the boundary of the 2-cell described by w_i, that overlap with an element of $\mathbb{Z}[t] \cdot \overline{\gamma_j}$. $\square$

Example 4.9 (Fox calculus for the trefoil)**.** The workings of the Fox calculus is best learnt through an example. Let's return to the fundamental group of the trefoil, and see that it matches up with our previous calculation. Recall that the knot group is given by

$$\pi_1(M_K) = \langle \alpha, \beta \mid \alpha\beta\alpha\beta^{-1}\alpha^{-1}\beta^{-1} \rangle.$$

Abelianising, we get that

$$|w| = |\alpha| + |\beta| + |\alpha| - |\beta| - |\alpha| - |\beta| = 0 \implies |\alpha| = |\beta| =: t,$$

so that $|\pi_1(M_K)| \simeq \mathbb{Z}\langle t \rangle$ as expected. We can then calculate the map d_2 via Fox calculus, and we have that

$$\begin{aligned} d_\alpha w =& d_\alpha(\alpha\beta\alpha\beta^{-1}\alpha^{-1}\beta^{-1}) \\ =& 1 + |\alpha\beta| d_\alpha(\alpha) + |\alpha\beta\alpha\beta^{-1} d_\alpha(\alpha^{-1}) \\ =& 1 + t^2 + t^2(-1)t^{-1} \\ =& t^2 - t + 1 \end{aligned}$$

and similarly

$$\begin{aligned} d_\beta w =& d_\alpha(\alpha\beta\alpha\beta^{-1}\alpha^{-1}\beta^{-1}) \\ =& |\alpha| d_\beta(\beta) + |\alpha\beta\alpha| d_\beta(\beta^{-1}) + |\alpha\beta\alpha\beta^{-1}\alpha^{-1}| d_\beta(\beta^{-1}) \\ =& t + t^3(-1)t^{-1} + t(-1)t^{-1} \\ =& -t^2 + t - 1, \end{aligned}$$

which is exactly what we calculated previously.

Definition 4.10 (Alexander matrix)**.** The matrix obtained via Fox calculus for d_2 is called the *Alexander matrix*, which we denote by A.

Analysing this matrix via module representations, it turns out that the Alexander matrix is also a presentation matrix for $H_1(\overline{M_K})$. For a deeper investigation into this, see chapter 6 and p.117 of Lickorish [**?**]

Proposition 4.11. Let K be a knot with knot group

$$\pi_1(M_K) = \langle \gamma_1, ..., \gamma_n \mid w_1, ..., w_{n-1} \rangle,$$

then $\Delta_K(t) \sim \det(A_n)$, where A_n is the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the last row.

Proof. The Wirtinger presentation gives us a group presentation with $n - 1$ relations, and so we have that our chain complex is of the form

$$0 \longrightarrow R^{n-1} \xrightarrow{A} R^n \xrightarrow{d_1} R \longrightarrow 0$$

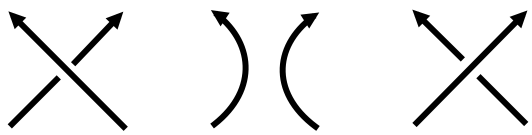
In particular, we know that since K is a knot, $|\gamma_i| = t$ for all i under the abelianisation map, and so $\ker(d_1) = \{(\gamma_1, ..., \gamma_n) \mid \sum \gamma_i = 0\}$. It follows then that the projection map $\pi : \ker(d_1) \to R^{n-1}$ is an isomorphism, so that

$$H_1(\overline{M_K}) = {}^{\ker(d_1)}\big/_{\mathrm{Im}(d_2)} \simeq {}^{R^{n-1}}\big/_{\mathrm{Im}(\pi \circ A)} \simeq {}^{R^{n-1}}\big/_{A_m},$$

so that $\Delta_K(t) \sim \det(A_m)$. ∎

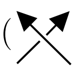
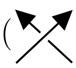
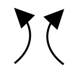
5 Properties of the Alexander polynomial

Having defined the Alexander polynomial, we will conclude this exposition by examining its properties, along with some of its advantages and disadvantages in comparison to other invariants. Perhaps the most convenient property of the Alexander polynomial is that it satisfies what is known as the *Conway skein relation*. Given a knot K, it is possible to unknot it by performing a sequence of crossing changes. The possible changes we can make to a crossing are



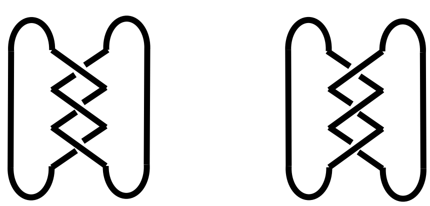
K_- $\quad$ K_0 $\quad$ K_+

The Conway Skein relation says that

$\Delta($  $) - \Delta($  $) = \left(t^{1/2} - t^{-1/2}\right) \Delta($  $)$

So that we can compute the Alexander polynomial of a knot starting with the Alexander polynomial of the unknot. The proof of this relies on understanding $\Delta_K(t)$ from the perspective of *Seifert surfaces* of K. For an in depth discussion of this, see chapter 6 of Lickorish [Lic97]. Thinking about the Alexander polynomial through this lens also allows us to define the *symmetrised Alexander polynomial*, which confirms that $\Delta_K(t)$ is not just a rational Laurent polynomial but an integral one, is (as the name suggests) symmetric, and satisfies $\Delta_K(1) = 1$. The Alexander polynomial is also easily generalisable - it can be extended from knots to links by a method analogous to the above constructions.

Though $\Delta_K(t)$ is a practical tool, it also has significant drawbacks. Of course, the Alexander polynomial is an easily computable and fundamental invariant of a knot, as it arises simply from consideration of the knot group. However, due to this dependence, it inherits negative properties of $\pi_1(M_K)$ that we wish to avoid. That is to say, it fails to capture a lot of characteristics of K we may wish to identify. For example, perhaps we're given the following two knots, the negative trefoil and the positive trefoil:



(a) $\qquad$ (b)

Figure 5: Knot diagram of (a) the negative trefoil and (b) the positive trefoil

Notice the difference in the crossings between the two. Suppose we'd like to distinguish between them. These two knots are not the same, but $\pi_1(M_K)$ fails to recognise that. In fact, for any knot K, its knot group is isomorphic to the knot group of its *mirror*, the knot described by reflecting K's knot diagram. This follows from since any knot is orientation reversing homeomorphic to its mirror, which induces an isomorphism on homotopy. Hence, for any knot K, $\Delta_K(t) \sim \Delta_{\overline{K}}(t)$, so

the Alexander polynomial also fails to distinguish between mirrors. Even worse, its not just true that two distinct knots will have distinct Alexander polynomials, but actually there are infinitely many knots that share an Alexander polynomial. There are many papers that prove this result, one of which is due to Kauffman [KL17]. So $\Delta_K(t)$ is certainly not a complete invariant.
Redeemingly, the Alexander polynomial plays a prominent part in modern knot theory and applicable areas. Seifert proved that the degree of the Alexander polynomial gives a lower bound for the minimal genus of a Seifert surface, the exact values of which is an NP-problem [AHT02]. For a proof of this bound, see [Ras21][Thm. 2.6.7]. Alexander polynomials also appear in knot Floer homology, as well as Seiberg-Witten theory. For further reading, see Ozsváth and Szábo's *An introduction to Heegaard Floer Homology* [OS06], and Rasmussen's *Floer homology and knot complements* [Ras03].

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